EXTENSION OF BERNSTEIN'S THEOREM TO STURM-LIOUVILLE SUMS*

BY

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One of the most important of recent theorems in analysis is a theorem due to S. Bernstein, which may be stated as follows:

If $T_n(x)$ is a trigonometric sum of order n, the maximum of whose absolute value does not exceed L, then the maximum of the absolute value of the derivative $T_n'(x)$ does not exceed nL.

Bernstein† proved the corresponding theorem for polynomials first, and from it obtained the theorem for the trigonometric case. His conclusion was that $|T_n'(x)|$ could not be so great as $2nL$. Various proofs were given by later writers,‡ leading to the simplified statement which appears above. The simplest proof was discovered independently by Marcel Riess§ and de la Vallée Poussin.||

The purpose of this paper is to prove the corresponding theorem for Sturm-Liouville sums:

The maximum of the absolute value of the derivative of a Sturm-Liouville sum of order $n (n \geq 1)$ can not exceed npM, where M is the maximum of the absolute value of the sum itself, and p is independent of n and of the coefficients in the sum.

The proof to be given here is similar to one which de la Vallée Poussin¶

* Presented to the Society, September 7, 1922.

† S. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné,* Mémoire couronné, Brussels, 1912, pp. 6–11 and 17–20.

‡ See, e. g., M. Riess, *Formule d'interpolation pour la dérivée d'un polynome trigonométrique,* Comptes Rendus, vol. 158 (1914), pp. 1152–1154; also F. Riess, *Sur les polynomes trigonométriques,* Comptes Rendus, vol. 158 (1914), pp. 1657–1661; and M. Fekete, *Über einen Satz von Serge Bernstein,* Journal für die reine und angewandte Mathematik, vol. 146 (1916), pp. 86–94.

§ M. Riess, *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome,* Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), p. 360.

|| C. de la Vallée Poussin, *Leçons sur l'Approximation des Fonctions d'une Variable réelle,* Paris, 1919, pp. 39–42; de la Vallée Poussin, *Sur le maximum du module de la dérivée d'une expression trigonométrique d'ordre et de module bornés,* Comptes Rendus, vol. 166 (1918), pp. 843–846.

¶ De la Vallée Poussin, op. cit., pp. 37–39. This proof gives the theorem in the less precise form $|T_n'(x)| \leqq npL$, where p is an absolute constant greater than unity.

gives for Bernstein's theorem. The simplest proof for the trigonometric case, to which reference was made above, seems not to be so readily carried over to the present problem.

Consider the system consisting of the differential equation

$$v'' + (\lambda + l(x))\, v = 0 \tag{1}$$

and the boundary conditions

$$v'(0) - h\,v(0) = 0, \qquad v'(\pi) + H\,v(\pi) = 0. \tag{2}$$

The function $l(x)$ is assumed to be continuous and to have continuous first and second derivatives in $0 \leqq x \leqq \pi$. The constants h and H are not restricted as to sign.

The characteristic numbers of this system are all real, and they can be arranged in a sequence, $\lambda_0, \lambda_1, \lambda_2, \ldots$, which has $+\infty$ as its only limit point, and is such that the characteristic solution corresponding to λ_k has exactly k zeros* in the interval from 0 to π. Not more than a finite number of the characteristic values λ_k can be negative, hence there is a negative number $-N$ such that $\lambda_k > -N$ for all values of k. From this it follows that we can rewrite the differential equation (1) in the form

$$v'' + (\varrho^2 + g(x))\, v = 0, \tag{3}$$

where

$$\varrho^2 = \lambda + N, \qquad g(x) = l(x) - N,$$

so that all the characteristic numbers λ_k correspond to positive real values of ϱ^2. The function $g(x)$ of course satisfies the conditions that were imposed on $l(x)$. If the positive square root of $\lambda_k + N$ is denoted by ϱ_k, all the numbers ϱ_k are real and greater than zero.

Asymptotic expressions† for the characteristic solutions and characteristic numbers of the differential equation (3) and the boundary conditions (2) are given by the equations

$$v_k(x) = \cos \varrho_k x + \frac{h}{\varrho_k} \sin \varrho_k x + \frac{1}{\varrho_k} \int_0^x g(t)\, v_k(t) \sin \varrho_k (t - x)\, dt, \tag{4}$$

$$\varrho_k = k + \varepsilon_k, \tag{5}$$

* M. Bôcher, *Leçons sur les Méthodes de Sturm,* Paris, 1917, p. 69.

† Cf. A. Kneser, *Untersuchungen über die Darstellung willkürlicher Funktionen in der mathemathischen Physik,* Mathematische Annalen, vol. 58 (1904), p. 120.

where $\lim_{k=\infty} \varepsilon_k = 0$, and, more precisely,*

$$|\varepsilon_k| < b_1/(k+1), \tag{6}$$

the quantity b_1 being independent of k. It is known furthermore that the functions $|v_k(x)|$ are uniformly bounded† for all values of k.

The main theorem to be established is an almost immediate consequence of the following, which we shall prove first.

Let $f(x)$ be an arbitrary bounded and measurable function in the interval $0 \leqq x \leqq \pi$. Let

$$S_n(x) = a_0 v_0 + a_1 v_1 + \cdots + a_{n-1} v_{n-1},$$

where the a's are the Sturm-Liouville coefficients for $f(x)$, defined by the formulas

$$a_k = \frac{1}{D_k} \int_0^\pi f(t)\, v_k(t)\, dt, \qquad D_k = \int_0^\pi v_k^2(t)\, dt. \tag{7}$$

Let

$$\sigma_n = \frac{S_1 + S_2 + \cdots + S_n}{n}.$$

If $|f(x)| \leqq M$ *throughout the interval, then* $|\sigma_n'(x)| \leqq nMG$, *where* G *is independent of* x, n, *and the choice of the function* $f(x)$.

For convenience, we shall define

$$S_{n1}(x) = \sum_{k=0}^{n-1} a_k \cos kx,$$

$$S_{n2}(x) = \sum_{k=0}^{n-1} a_k [\cos \varrho_k x - \cos kx],$$

$$S_{n3}(x) = \sum_{k=0}^{n-1} a_k \frac{h}{\varrho_k} \sin \varrho_k x,$$

$$S_{n4}(x) = \sum_{k=0}^{n-1} \frac{a_k}{\varrho_k} \int_0^x g(t)\, v_k(t) \sin \varrho_k (t - x)\, dt.$$

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* The letter b with subscripts is used throughout to denote constants independent of x, k, n, and the function $f(x)$ which presently enters into the discussion. We shall write $b/(k+1)$ rather than b/k in various places in order that the formulas may be accurate even if $k = 0$.

† Cf. Kneser, loc. cit., p. 118.

Then we can write

$$S_n(x) = S_{n1}(x) + S_{n2}(x) + S_{n3}(x) + S_{n4}(x),$$

$$\sigma_n(x) = \sigma_{n1}(x) + \sigma_{n2}(x) + \sigma_{n3}(x) + \sigma_{n4}(x),$$

where

$$\sigma_{ni} = \frac{S_{1i} + S_{2i} + \cdots + S_{ni}}{n} \qquad (i = 1, 2, 3, 4).$$

To prove the preliminary theorem stated, we shall show that

$$|\sigma'_{ni}(x)| \leqq n M G_i \qquad (i = 1, 2, 3, 4),$$

each G_i being a constant of the same character as the G mentioned above.

Let us first prove that $|a_k| \leqq M b_2$ for all values of k.

From (4) and the fact that v_k is bounded, it follows that we may write*

$$v_k(x) = \cos \varrho_k x + \frac{r_1}{k+1}$$

$$= \cos kx + [\cos \varrho_k x - \cos kx] + \frac{r_1}{k+1}.$$

Now, from (5),

$$\cos \varrho_k x - \cos kx = -2 \sin\left(\frac{1}{2}\, \varepsilon_k x\right) \sin\left[\left(k + \frac{1}{2}\, \varepsilon_k\right) x\right].$$

Since

$$\left|\sin\left(\frac{1}{2}\, \varepsilon_k x\right)\right| \leqq \left|\frac{1}{2}\, \varepsilon_k x\right| \text{ and } 0 \leqq x \leqq \pi,$$

it follows from (6) that

(8) $$\left|\sin \frac{\varepsilon_k x}{2}\right| < \frac{b_1 \pi}{2(k+1)}$$

and $\cos \varrho_k x - \cos kx = r_2/(k+1)$. Hence we have

(9) $$v_k(x) = \cos kx + \frac{r_3}{k+1}.$$

* The letter r with subscripts is used to denote functions of x which may depend on the subscript k, but are uniformly bounded for all values of k.

Then

$$v_k^2(x) = \cos^2 kx + \frac{r_4}{k+1},$$

and

$$D_k = \int_0^\pi v_k^2(t)\,dt = \int_0^\pi \cos^2 kt\,dt + \frac{1}{k+1}\int_0^\pi r_4\,dt = \frac{\pi}{2} + \gamma_k, \quad |\gamma_k| < \frac{b_3}{k+1}.$$

Therefore

$$\frac{1}{D_k} = \frac{2}{\pi} + \gamma_k', \qquad |\gamma_k'| < \frac{b_4}{k+1}, \tag{10}$$

so that the positive quantity $1/D_k$ is less than some constant b_5. The other factor in the expression (7) for a_k is less than or equal to Mb_6 in absolute value, since $|f(x)| \leqq M$ and $v_k(x)$ is uniformly bounded.

Consequently

$$|a_k| \leqq M b_6 b_5 = M b_2.$$

Now let us consider the expression $\sigma_{n2}'(x)$. The general term of S_{n2}, apart from the constant coefficient, is $\cos \varrho_k x - \cos kx$, which has for its derivative

$$[\cos \varrho_k x - \cos kx]'$$
$$= -\varepsilon_k \cos \frac{\varepsilon_k x}{2} \sin\left(k + \frac{\varepsilon_k}{2}\right)x - (2k + \varepsilon_k) \sin \frac{\varepsilon_k x}{2} \cos\left(k + \frac{\varepsilon_k}{2}\right)x.$$

From (6) and (8) it follows that

$$|[\cos \varrho_k x - \cos kx]'| < \frac{b_1}{k+1} + (2k + \varepsilon_k)\frac{b_1 \pi}{2(k+1)} < b_7$$

for all k. Hence

$$|S_{n2}'(x)| \leqq b_7 \sum_{k=0}^{n-1} |a_k| \leqq n b_7 M b_2 = n M G_2.$$

Now

$$\sigma_{n2}'(x) = \frac{S_{12}' + S_{22}' + \cdots + S_{n2}'}{n}$$

and therefore

$$|\sigma'_{n2}(x)| \leqq \frac{MG_2 + 2MG_2 + \cdots + nMG_2}{n} \leqq nMG_2.$$

From the definition of $S_{n3}(x)$ and the fact that $|a_k| \leqq Mb_2$, it is seen that

$$|S'_{n3}(x)| = \left|\sum_{k=0}^{n-1} h\, a_k \cos \varrho_k x\right| \leqq nMb_2 h = nMG_3.$$

By the same argument as used above, it follows that

$$|\sigma'_{n3}(x)| \leqq nMG_3.$$

The derivative of $S_{n4}(x)$ contains only terms of the form

$$-a_k \int_0^x g(t)\, v_k(t) \cos \varrho_k(t-x)\, dt,$$

for the terms resulting from the differentiation with respect to the upper limit of integration all reduce to zero. Each term of $S'_{n4}(x)$ is in absolute value less than or equal to Mb_8, since the integrand is uniformly bounded and $|a_k| \leqq Mb_2$ for all values of k. Consequently

$$|S'_{n4}(x)| \leqq nMG_4,$$

and, as a result,

$$|\sigma'_{n4}(x)| \leqq nMG_4.$$

It remains to prove that $|\sigma'_{n1}(x)| \leqq nMG_1$.

To do this it is necessary to ascertain the magnitude of a_k more accurately. This can be accomplished by substituting in the formula for a_k the expression for $v_k(t)$ given by (9). Thus

$$a_k = \frac{1}{D_k} \int_0^\pi f(t) \left[\cos kt + \frac{r_3}{k+1}\right] dt,$$

which, by application of (10), reduces to

$$a_k = \frac{2}{\pi} \int_0^\pi f(t) \cos kt\, dt + \frac{1}{k+1} \int_0^\pi f(t)\, r_5(t)\, dt.$$

Substituting this value for a_k in the expression for $S_{n1}(x)$, we have

$$S_{n1}(x) = \sum_{k=0}^{n-1} \frac{2}{\pi} \int_0^{\pi} f(t) \cos kt \cos kx \, dt + \sum_{k=0}^{n-1} \frac{\cos kx}{k+1} \int_0^{\pi} f(t) r_5(t) \, dt.$$

Let these two sums be denoted by $\bar{S}_{n1}$ and $\bar{\bar{S}}_{n1}$, and the corresponding means by $\bar{\sigma}_{n1}$ and $\bar{\bar{\sigma}}_{n1}$, so that $\bar{\sigma}_{n1} + \bar{\bar{\sigma}}_{n1} = \sigma_n$. Then

$$\bar{\bar{S}}'_{n1} = \sum_{k=0}^{n-1} \frac{-k \sin kx}{k+1} \int_0^{\pi} f(t) r_5(t) \, dt$$

and

$$|\bar{\bar{S}}'_{n1}| \leqq \sum_{k=0}^{n-1} M b_9 = n M b_9, \qquad |\bar{\bar{\sigma}}'_{n1}| \leqq n M b_9.$$

To prove that $|\bar{\sigma}'_{n1}| \leqq n M b_{10}$, we need the explicit form for $\bar{\sigma}_{n1}(x)$. Inasmuch as

$$\bar{\sigma}_{n1} = \frac{\bar{S}_{11} + \bar{S}_{21} + \cdots + \bar{S}_{n1}}{n}$$

and

$$\bar{S}_{n1} = \sum_{k=0}^{n-1} \frac{2}{\pi} \int_0^{\pi} f(t) \cos kt \cos kx \, dt,$$

it is seen that

$$\bar{\sigma}_{n1} = \frac{2}{\pi n} \int_0^{\pi} f(t) \, [n + (n-1) \cos t \cos x + (n-2) \cos 2t \cos 2x + \cdots \\ \cdots + \cos (n-1)t \cos (n-1)x] \, dt$$

$$= \frac{1}{\pi n} \int_0^{\pi} f(t) \left[n + \frac{\sin^2 n \left(\frac{x+t}{2}\right)}{2 \sin^2 \left(\frac{x+t}{2}\right)} + \frac{\sin^2 n \left(\frac{x-t}{2}\right)}{2 \sin^2 \left(\frac{x-t}{2}\right)} \right] dt.$$

Since (as appears from the cosine expression) the integrand is continuous in x and t and has a continuous derivative with respect to x, the conditions for differentiation under the integral sign are satisfied, and we have

$$\bar{\sigma}'_{n1} = \frac{1}{\pi n}\int_0^\pi f(t)\left[\frac{\partial}{\partial x}\frac{\sin^2 n\left(\frac{x+t}{2}\right)}{2\sin^2\left(\frac{x+t}{2}\right)} + \frac{\partial}{\partial x}\frac{\sin^2 n\left(\frac{x-t}{2}\right)}{2\sin^2\left(\frac{x-t}{2}\right)}\right]dt.$$

From this it follows that

$$|\bar{\sigma}'_{n1}| \leqq \frac{M}{\pi n}\left[\int_0^\pi \left|\frac{\partial}{\partial x}\frac{\sin^2 n\left(\frac{x+t}{2}\right)}{2\sin^2\left(\frac{x+t}{2}\right)}\right| dt + \int_0^\pi \left|\frac{\partial}{\partial x}\frac{\sin^2 n\left(\frac{x-t}{2}\right)}{2\sin^2\left(\frac{x-t}{2}\right)}\right| dt\right].$$

In the first of these integrals, let $\frac{1}{2}(x+t) = u$, and in the second, let $\frac{1}{2}(x-t) = u$; in each case, $\partial/\partial x = \frac{1}{2}(d/du)$. Making these substitutions, we have

$$|\bar{\sigma}'_{n1}| \leqq \frac{M}{\pi n}\left[\int_{x/2}^{(x/2)+(\pi/2)} \left|\frac{d}{du}\frac{\sin^2 nu}{2\sin^2 u}\right| du - \int_{x/2}^{(x/2)-(\pi/2)} \left|\frac{d}{du}\frac{\sin^2 nu}{2\sin^2 u}\right| du\right].$$

Now the two integrals have the same integrand. Moreover, if the limits of integration of the second integral be reversed and the sign changed to compensate, then the two integrals can be combined into one integral over the interval from $\frac{1}{2}(x-\pi)$ to $\frac{1}{2}(x+\pi)$. Since the integrand is of period π, this interval can be replaced by that from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$. Furthermore, the integrand is an even function; hence the integral can be replaced by twice the integral from 0 to $\frac{1}{2}\pi$. Thus the inequality becomes

$$|\bar{\sigma}'_{n1}| \leqq \frac{M}{\pi n}\int_0^{\pi/2} \left|\frac{d}{du}\frac{\sin^2 nu}{\sin^2 u}\right| du.$$

The integral last written down is equal to the total variation of the function $\psi(u)$ in the interval from 0 to $\frac{1}{2}\pi$, where $\psi(u)$ is the continuous function defined by the relations

$$\psi(u) = \frac{\sin^2 nu}{\sin^2 u}, \qquad 0 < u \leqq \frac{\pi}{2},$$

$$\psi(0) = n^2.$$

In order to determine the value of the total variation, let us study the graph of $\psi(u)$ in $(0, \frac{1}{2}\pi)$. It may be assumed that $n>1$. The function is equal to zero at the points $u = q\pi/n$, $q = 1, 2, \ldots, n_1$, where n_1 stands for the greatest integer contained in $\frac{1}{2}n$. Its derivative (for $u > 0$) is

$$\psi'(u) = \frac{2 \sin nu}{\sin u} \cdot \frac{n \sin u \cos nu - \sin nu \cos u}{\sin^2 u}.$$

Hence $\psi(u)$ can have a maximum or minimum only at the points $u = q\pi/n$, and at the points where $\Phi(u) = n \sin u \cos nu - \sin nu \cos u$ vanishes.

In any one of the intervals $q\pi/n \leqq u \leqq (q+1)\pi/n$, $\Phi(u)$ has only one zero. If $\Phi(u)$ had two zeros in one of these intervals, then $\Phi'(u) = (1-n^2)\sin nu \sin u$ would have to vanish in the interior of the interval. But $\Phi'(u)$ vanishes only at the ends of the interval. Furthermore, $\psi'(u)$ must have one zero in each interval, for $\psi(q\pi/n) = \psi((q+1)\pi/n) = 0$.

In $0 \leqq u \leqq \pi/n$, $\Phi(u)$ vanishes only at $u = 0$. If $\Phi(u)$ had a zero at u_1 interior to the interval, then $\Phi'(u)$ would have a zero between 0 and u_1, which is impossible.

The function $\psi(u)$, then, has a maximum at $u = 0$, a minimum at each of its zero points, $u = q\pi/n$, and just one maximum in each of the intervals $q\pi/n \leqq u \leqq (q+1)\pi/n$.

In $0 < u < \frac{1}{2}\pi$, $(\sin u)/u > (\sin \frac{1}{2}\pi)/(\frac{1}{2}\pi) = 2/\pi$, hence $\sin u > (2/\pi) \cdot (q\pi/n) = 2q/n$ throughout the interval $q\pi/n \leqq u \leqq (q+1)\pi/n$. From this inequality and from the fact that $\sin^2 nu \leqq 1$, it follows that the maximum of $\psi(u)$ in this interval is less than $n^2/(4q^2)$ and hence the total variation of $\psi(u)$ in the interval is less than $n^2/(2q^2)$. In the interval $0 \leqq u \leqq \pi/n$, the value of $\psi(u)$ descends from the maximum n^2 to zero, and the total variation is simply n^2. For the whole interval from 0 to $\frac{1}{2}\pi$, then, the total variation of $\psi(u)$ is less than

$$n^2\left[1 + \frac{1}{2}\left(1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n_1^2}\right)\right],$$

which is less than a quantity of the form $b_{11}n^2$, since the parenthesis is the sum of a finite number of terms of a positive convergent series. Therefore,

$$|\bar{\sigma}'_{n1}| \leqq b_{11}n^2 \cdot \frac{M}{\pi n} = \frac{b_{11}}{\pi} nM = nMb_{10}.$$

Since $\sigma_{n1} = \bar{\sigma}_{n1} + \bar{\bar{\sigma}}_{n1}$, it follows that

$$|\sigma'_{n1}| \leqq nMb_9 + nMb_{10} = nMG_1.$$

By combination of this inequality with those previously obtained, it is seen that

$$|\sigma'_n(x)| \leqq nMG_1 + nMG_2 + nMG_3 + nMG_4,$$

which is equivalent to the desired relation

$$|\sigma'_n(x)| \leqq nMG.$$

We are now ready to prove the main theorem of the paper, the extension of Bernstein's theorem to Sturm-Liouville sums. The preceding work will be applied by allowing $f(x)$ itself to be such a sum. Let $S_n(x)$ be an arbitrary Sturm-Liouville sum of order $n-1$,

$$S_n(x) = a_0 v_0(x) + a_1 v_1(x) + \cdots + a_{n-1} v_{n-1}(x),$$

and M the maximum of its absolute value for $0 \leqq x \leqq \pi$.

To prove the theorem as stated, we should show that $|S'_n(x)| \leqq (n-1)pM$. It is sufficient, however, apart from a change in the numerical value of p, to prove that $|S'_n(x)| \leqq npM$, for if p' is taken equal to $2p$, $npM \leqq (n-1)p'M$ when $n > 1$. If $n = 1$, $S_n(x) = a_0 v_0$; that is, the sum is of order zero, and for this case the theorem does not hold in general.

Let the notation of the previous work be used, with $f(x) = S_n(x)$, as already suggested. By the definition of the quantities σ,

$$\sigma_{2n} = \frac{S_1 + S_2 + \cdots + S_{2n}}{2n}.$$

But as $f(x)$ is a Sturm-Liouville sum of order $n-1$, it is identical with the partial sum of its own Sturm-Liouville expansion to terms of the $(n-1)$st order. That is,

$$S_i = S_n \text{ if } i \geqq n,$$

and

$$\sigma_{2n} = \frac{S_1 + S_2 + \cdots + S_n + nS_n}{2n} = \frac{1}{2}\sigma_n + \frac{1}{2}S_n,$$

whence*

$$S_n = 2\sigma_{2n} - \sigma_n.$$

Therefore we can write

$$S_n'(x) = 2\sigma_{2n}'(x) - \sigma_n'(x)$$

and

$$|S_n'(x)| \leqq 2\,|2\sigma_{2n}'(x)| + |\sigma_n'(x)| \leqq 4\,nMG + nMG = npM$$

where p is a constant independent of x, n, and the coefficients in $S_n(x)$.

* Cf. de la Vallée Poussin, op. cit., p. 33.

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